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ON FINITE DEFORMATION ELASTO-PLASTICITY

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Abstract—Some fundamental kinematical and kinetical results in finite elasto-plastic deformations of crystalline solids are reviewed. It is shown that essentially all existing elasto-plasticity concepts lead rigorously to total strain rate measures which are additively decomposed to an elastic and a plastic contribution, provided that the corresponding total, elastic and plastic strain rates are conjugate to the *same* stress measure. Furthermore, this additivity follows from the conservation of energy which further shows that, under the most general setting, the plastic strain rate may include a "workless" additive part which renders the Eulerian plastic strain rate tensor noncoaxial with the Cauchy stress tensor even when an isotropic yield function is assumed. Various Lagrangian and Eulerian strain measures, their rates and the corresponding conjugate stress measures are examined, and it is established that the additive decomposition of the strain rate holds independently of the particular choice of the strain measure or the ground state. Finally, a conflicting theory by Lee[33, 36], who claims to have shown that the usual additive decomposition of the strain rate to an elastic and a platic part is in "error", is reviewed, and it is shown that this theory also leads to an additive strain rate decomposition, and that Lee's conflicting conclusion stems from misinterpretation. Certain undesirable features of this theory, which emerge from a decomposition of the "total" deformation gradient into an elastic and a plastic part, are discussed and it is concluded that the commonly accepted physically based incremental theories of elasto-plasticity, either phenomenological or microscopically based, present distinct advantages.

1. INTRODUCTION

Rate-independent elasto-plastic, *infinitesimal* deformations of crystalline solids involve strain rates which are additively decomposed to an elastic and a plastic part. This holds whether the solid is isotropic or anisotropic, homogeneous or heterogeneous. It forms the basis of both phenomenological macroscopic and microscopic (small strain) plasticity theories.

It follows from these remarks that *infinitesimal incremental* deformations from a given *finitely* deformed configuration of an elasto-plastic solid must also satisfy the additivity, *provided that the same measure of strain increment with respect to the same configuration is used to define the total, the elastic and the plastic strain increments.* Hence, the Eulerian strain rate, **D**, which is the symmetric part of the velocity gradient with respect to the *current* configuration, decomposes as

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \tag{1.1}$$

where superscripts e and p refer to the elastic and plastic contribution, respectively.

The decomposition (1.1) permits construction of rate constitutive relations for elasto-plastic crystalline solids, where plasticity stems from slip over crystallographic slip planes, and the elasticity arises from lattice distortion which produces a compatible overall rate of deformation. For single crystals, micromechanical modeling of this kind has been successfully implemented for small strains, as well as arbitrary finite deformations[1–9].

Based on the slip theory of single crystals, models have been developed to describe the overall mechanical behavior of polycrystals, beginning with the classic work of Taylor[10, 11]. Because of basic mathematical difficulties, essentially all calculations for this class of problems have been for small strain elasto-plasticity, although Hill's[12] fundamental averaging theorems do provide a general framework for finite deformations; see, e.g. [4, 11, 13–29], where references to related work can be found. Models used by different researchers vary in a number of aspects which include the basic slip-induced hardening assumptions and the method by which the local kinematical and dynamical quantities are estimated in terms of the corresponding overall macroscopic values. Various hardening laws have been discussed and compared by Hill[3], and

recently further expounded upon by Havner[30]. In [22], Hutchinson gives a detailed illustrative account of three major averaging techniques attributed to Lin[16], Kröner[18], Budiansky and Wu[19], and Hill[31], and shows that Hill's self-consistent model provides the least stiff overall response.

Recently, with my graduate student, Mr. T. Iwakuma, I[32] formulated in the context of Hill's[31], as well as Kröner–Budiansky–Wu's[18, 19] self-consistent models, solution procedures for obtaining the overall macroscopic instantaneous moduli in terms of the slip-induced elasto-plasticity of single crystals for arbitrary finite elasto-plastic deformations. Since the rate problem can be cast in terms of the nominal stress rate, Hill's averaging theorems[12] play a prominent role in this formulation. Lack of usual symmetry, and dependence on overall stress state render the calculation of the needed Green's functions a difficult task which, however, can be performed for specific problems such as those involved in uniaxial and biaxial loading regimes.

All the above-mentioned developments in elasto-plasticity are based on the fundamental decomposition (1.1).

In an interesting paper, Lee[33] proposes a theory based on the decomposition of the *total* deformation gradient to an elastic and plastic part, and from it concludes that the additive decomposition of the strain rates, (1.1), holds only approximately. Some of the difficulties involved in Lee's arguments which might have led to his rather surprising conclusion, have been examined recently by Nemat-Nasser[34] and Mandel[35]. The analysis in [34] has apparently not been convincing enough, because two other papers[36, 37] authored or coauthored by Prof. Lee have since appeared, in which further arguments in support of nonadditivity of strain rates are presented. Indeed, Lubarda and Lee in [37] declare that Lee's[33] "exact finite-deformation kinematics shows the almost universal assumption that the total velocity strain or rate of deformation is the sum of elastic and plastic rates to be in error". Hence, question is raised on the validity of essentially all finite deformation elasto-plasticity theories.

To bring this question to a satisfactory resolution, in Section 2, I shall first give a concise account of a general theory of finite elasto-plasticity presented by Hill and Rice[38], Mandel[39] and Hill[40], tie it to the commonly accepted slip theory of crystalline solids, and show that the additive decomposition of the strain rates follows rigorously from essentially all common finite elasto-plasticity concepts. Then I shall examine Lee's theory, and show that this theory also leads to an additive strain rate decomposition, and therefore, his conclusion stems from misinterpretation. Since, in finite deformations, various strain rates may be used, and since Lee's theory envisages at each instant three distinct configurations, namely the initial, the current, and an intermediate (unstressed) one, it turns out that the elastic and the plastic strain rates that Lee considers *do not correspond to the same configuration*, and are, therefore, not compatible measures; i.e. they are *not* addable quantities. This is indeed the cause of Lee's unexpected conclusion.

I shall argue that the elastic, the plastic and the total strain rates follow the additivity rigorously, *provided that they are conjugate to the same stress measure.* This is based on the observation, Hill[40], that the quantity $1/\rho_0 \operatorname{tr}(\boldsymbol{\tau}\mathbf{D}) = 1/\rho_0 \tau_{ij} D_{ji}$, $i, j = 1, 2, 3$, is the rate of work per unit mass, and is invariant with respect to the change of the deformation measure (at least within a suitable class) and the reference state; here ρ_0 is a reference mass density, $\mathbf{D}$ is the Eulerian strain rate conjugate to the Kirchhoff stress $\boldsymbol{\tau}$, D_{ij} and τ_{ij} are the corresponding components with respect to a fixed rectangular Cartesian coordinate system, and repeated indices are summed. Thus, if $\mathbf{D}^e$ and $\mathbf{D}^p$ are conjugate to the *same* stress measure $\boldsymbol{\tau}$ referred to the *same* ground state of mass density ρ_0, then by necessity, i.e. in view of the conservation of energy, it follows that

$$\frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}) = \frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}^e) + \frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}^p) \tag{1.2}$$

which is consistent with (1.1). If in (1.2) a different *common* stress measure is used, then the strain rate measures will change accordingly, but *additively.*

The choice of a particular strain measure does not alter the physical content of (1.2) nor,

therefore, the validity of (1.1) or its equivalents. Each quantity in (1.2) represents the corresponding rate of energy per unit mass and, hence, remains invariant with respect to the change of strain measure or the reference state. Suppose, for example, that $\hat{\mathbf{D}}^p$ is the plastic part of the deformation rate tensor referred to another reference state of mass density $\hat{\rho}$, and let $\hat{\tau}$ be its conjugate stress, so that

$$\frac{1}{\rho_0}\operatorname{tr}(\tau\mathbf{D}^p)=\frac{1}{\hat{\rho}}\operatorname{tr}(\hat{\tau}\hat{\mathbf{D}}^p). \tag{1.3}$$

Then, naturally, the last term in (1.2) is precisely equivalent to the r.h.s. of (1.3), and if one chooses to use $\hat{\mathbf{D}}^p$ as the plastic part of the strain rate and $\mathbf{D}^e$ as the elastic part, they will not add up to $\mathbf{D}$. For other reasons, it may be more convenient to choose $\hat{\mathbf{D}}^p$ for the plastic part of the strain rate, but this choice is essentially arbitrary, and does not alter the validity of (1.1) or its equivalent.

The decomposition (1.1) not only follows from the physics of elasto-plasticity, but also can be "proved" on the basis of the conservation of energy, as briefly discussed below.

1.1 *An energy-based proof for additive decomposition of strain rates*

Let w be the rate of stress-work per unit mass, and let w^e and w^p, respectively, be the rate of energy associated with elastic distortion (elastic stored energy) and the rate of energy dissipated due to plastic flow. For example, in crystals, w^e is the rate of energy stored due to lattice distortion, and w^p is the rate of energy dissipated due to slip over crystallographic planes. In the absence of any other form of energy, the conservation of energy requires that

$$w = w^e + w^p. \tag{1.4}$$

On the other hand,

$$w=\frac{1}{\rho_0}\operatorname{tr}(\tau\mathbf{D}). \tag{1.5}$$

As the body deforms elasto-plastically, *both elastic distortion and plastic flow occur simultaneously under the action of the same existing overall true stress*, $\sigma=(\rho/\rho_0)\tau$. Therefore, one can write, in general,

$$w^e=\frac{1}{\rho_0}\operatorname{tr}(\tau\mathbf{y}),\quad w^p=\frac{1}{\rho_0}\operatorname{tr}(\tau\mathbf{z}), \tag{1.6}$$

where $\mathbf{y}$ and $\mathbf{z}$ are the (rate) kinematical quantities conjugate to the Kirchhoff stress τ, pertaining to, respectively, the elastic and the plastic distortions. Substitution from (1.5) and (1.6) into (1.4) now yields

$$\operatorname{tr}[\tau(\mathbf{D}-\mathbf{y}-\mathbf{z})]=0 \tag{1.7}$$

which has the following general solution,

$$\mathbf{D}=\mathbf{y}+\mathbf{z}+\mathbf{\Lambda}, \tag{1.8}$$

where $\mathbf{\Lambda}$ is the strain rate that produces *no* work, i.e.

$$\operatorname{tr}(\tau\mathbf{\Lambda})=0. \tag{1.9}$$

If $\mathbf{y}$ is identified with the elastic part of the deformation rate, and $\mathbf{z}+\mathbf{\Lambda}$ with the plastic part of the deformation rate, then eqn (1.1) is obtained.

It is interesting to note, therefore, that the plastic contribution to the total deformation rate, $\mathbf{D}^p$, may, in general, include a portion which does not contribute to the rate of plastic work, i.e.

a "workless" component. Since (1.9) can be expressed as

$$\mathrm{tr}\,(\boldsymbol{\tau}\boldsymbol{\Lambda}) = \frac{1}{3}\mathrm{tr}\,(\boldsymbol{\tau})\,\mathrm{tr}\,(\boldsymbol{\Lambda}) + \mathrm{tr}\,(\boldsymbol{\tau}'\boldsymbol{\Lambda}) = 0, \tag{1.10}$$

it is observed that $\boldsymbol{\Lambda}$ is deviatoric, $\mathrm{tr}\,(\boldsymbol{\Lambda}) = 0$ and (1.9) becomes

$$\mathrm{tr}\,(\boldsymbol{\tau}'\boldsymbol{\Lambda}') = 0, \tag{1.11}$$

where prime denotes the deviatoric part. A particular solution of eqn (1.11) is

$$\boldsymbol{\Lambda}' = A\left(\frac{\boldsymbol{\tau}'}{\bar{\tau}}\right)^{\nabla}, \quad \bar{\tau} = \mathrm{tr}\left(\frac{1}{2}\boldsymbol{\tau}'\boldsymbol{\tau}'\right), \tag{1.12}$$

where the superposed ∇ denotes a suitable objective time rate of change, and A is a rate-independent material parameter. Note that, since $\boldsymbol{\tau}'/\bar{\tau}$ is a constant tensor, its time rate is "normal" to itself. While (1.12) is not the most general solution to (1.11) for three-dimensional stress-states (it *is* for the two-dimensional ones), it has historical significance, as it emerges in a natural way in the double sliding theory of granular materials; see Mandel[41], de Josselin de Jong[42], Spencer[43], Mandl and Fernández Luque[44] and Mehrabadi and Cowin[45]. The same term, but with a different interpretation, also plays a prominent role in the plasticity theories considered by Rudnicki and Rice[46] and Stören and Rice[47]. More recently, by examining the micromechanical behavior of granular masses that consist of frictional rigid granules, Christoffersen *et al.*[48] have arrived at expressions which include precisely the same kind of workless plastic component.†

In the final part of the present paper I examine some of the physical implications of the basic hypothesis in Lee's theory which requires a *total* (as contrasted to an infinitesimal) elastic unloading from a current finitely deformed state to an intermediate plastically deformed state *without* additional plastic flow. Clearly, when finite elastic unloading is envisaged, *reverse plastic flow* becomes inevitable, and such a procedure cannot be implemented experimentally. The hypothesis, therefore, has only a conceptual value rather than an actual physical one. For this reason, its merits can only be assessed from a theoretical viewpoint. I shall show that Lee's theory leads to some undesirable features, and therefore the commonly accepted physically based incremental theories of elasto-plasticity, either phenomenological or microscopically based, present distinct advantages.

2. A GENERAL FINITE ELASTO-PLASTICITY THEORY WITH ADDITIVE STRAIN RATE DECOMPOSITION

Rate-independent incremental deformation of crystalline solids is usually envisioned to involve two accompanying microprocesses: (1) slip over instantaneously active crystallographic planes; and (2) accommodating lattice distortion. The first is viewed as the plastic, and the second as the elastic, contribution to the total incremental deformation. For infinitesimal increments, additivity follows rigorously from the physics for any appropriate material strain measures. In polycrystalline solids, accommodation is required to satisfy overall compatibility and continuity of deformation across grain boundaries. The entire matter can be concisely and elegantly displayed with the aid of a material strain measure, say, $\mathbf{E}$, e.g. the Lagrangian strain, and a measure of history dependence, collectively denoted by $\mathbf{H}$, as discussed by Hill and Rice‡[38] and Hill[40]; this theory covers both rate-dependent and rate-independent materials, but here only rate-independence is considered. Incremental deformations with history-measure $\mathbf{H}$ kept fixed at its current value are regarded as purely elastic, admitting Helmholtz free energy $\varphi = \varphi(\mathbf{E}, \mathbf{H})$ measured per unit volume of the chosen reference state, so that

†Note that even when an isotropic yield function is assumed, this workless plastic strain rate component makes the plastic strain rate tensor noncoaxial with the Cauchy stress tensor. For this reason, this term is often called "the noncoaxiality term".

‡See also Mandel[35, 39].

$$\mathbf{S} \equiv \frac{\partial \varphi}{\partial \mathbf{E}} \quad \text{or} \quad S_{AB} = \frac{\partial \varphi}{\partial E_{AB}}, \quad A, B = 1, 2, 3, \tag{2.1}$$

is the stress conjugate to the strain measure $\mathbf{E}$,

$$\operatorname{tr}(\mathbf{S}\, \mathrm{d}\mathbf{E}) \equiv S_{AB}\, \mathrm{d}E_{BA} = d^e\varphi, \quad \mathrm{d}\mathbf{H} = \mathbf{0}, \tag{2.2}$$

where repeated indices are summed, and S_{AB} and E_{AB}, $A, B = 1, 2, 3$, are the rectangular Cartesian components of $\mathbf{S}$ and $\mathbf{E}$, respectively; the operator d^e denotes the increment at constant $\mathbf{H}$. In the sequel both the direct and the Cartesian notation will be used.

Again at fixed $\mathbf{H}$, the complementary energy function $\psi = \psi(\mathbf{S}, \mathbf{H})$ relates to φ by the usual Legendre transformation,

$$\psi \equiv \frac{\partial \varphi}{\partial E_{AB}} E_{AB} - \varphi, \quad (\mathbf{H} \text{ fixed}), \tag{2.3}$$

and one has

$$\mathbf{E} \equiv \frac{\partial \psi}{\partial \mathbf{S}}; \tag{2.4}$$

above and throughout this paper, partial differentiation notation (round dee, ∂) is used when all other parameters in the argument of the function other than the one explicitly displayed, are kept fixed.

Consider now an infinitesimal elasto-plastic change which also results in the change of $\mathbf{H}$ to $\mathbf{H} + \mathrm{d}\mathbf{H}$. Then the *total* strain increment can be written as

$$\mathrm{d}\mathbf{E} = d^e\mathbf{E} + d^p\mathbf{E}, \tag{2.5}$$

where the elastic and plastic contributions to the total strain increment are respectively given by

$$d^e\mathbf{E} \equiv \frac{\partial^2 \psi}{\partial \mathbf{S} \partial S_{AB}}\, \mathrm{d}S_{AB}, \tag{2.6}$$

$$\begin{aligned} d^p\mathbf{E} &\equiv \mathbf{E}(\mathbf{S}, \mathbf{H} + \mathrm{d}\mathbf{H}) - \mathbf{E}(\mathbf{S}, \mathbf{H}), \\ &= \frac{\partial}{\partial \mathbf{S}} [\psi(\mathbf{S}, \mathbf{H} + \mathrm{d}\mathbf{H}) - \psi(\mathbf{S}, \mathbf{H})] = \frac{\partial}{\partial \mathbf{S}}\, d^p\psi. \end{aligned} \tag{2.7}$$

From (2.6) the symmetric fourth-order tensor

$$\mathscr{M}^0 \equiv \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \mathbf{S}} \quad \text{or} \quad \mathscr{M}^0_{ABCD} = \frac{\partial^2 \psi}{\partial S_{AB} \partial S_{CD}} \tag{2.8}$$

is the instantaneous elastic compliance for the strain measure $\mathbf{E}$ relative to the chosen reference configuration of mass density ρ_0. The instantaneous elastic compliance tensor changes with the change of the stress (and hence the strain) measure or the reference state, as discussed by Hill [49]. For example, if $\mathbf{E}$ is the usual Lagrangian strain, then when the reference configuration coincides with the current configuration (Eulerian description), it can be deduced from (2.6) that

$$\dot{\mathbf{E}}^e \equiv \mathbf{D}^e = \mathscr{M} : \overset{*}{\boldsymbol{\tau}} \quad \text{or} \quad D^e_{ij} = \mathscr{M}_{ijkl} \overset{*}{\tau}_{kl}, \tag{2.9}$$

where $\mathscr{M}$ with Cartesian components $\mathscr{M}_{ijkl}$ is the instantaneous *elastic* compliance tensor (having the usual symmetries), measured from the current stressed state at *the constant current current value of* $\mathbf{H}$; $\overset{*}{\boldsymbol{\tau}}$ is a stress rate corotational with the material which deforms at constant $\mathbf{H}$, that is

$$\overset{*}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W}^*\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{W}^*, \tag{2.10}$$

and if $\boldsymbol{\tau}$ stands for the Kirchhoff stress, then

$$\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \operatorname{tr} \mathbf{D}^e, \tag{2.11}$$

where $\mathbf{W}^*$ is the spin associated with elastic distortion (*spin at constant* $\mathbf{H}$), $\boldsymbol{\sigma}$ is the Cauchy stress (equals the Kirchhoff stress when the current configuration is used as the reference one), and $\operatorname{tr}\mathbf{D}^e$ is the rate of dilatation associated with the elastic deformation rate (i.e. at constant $\mathbf{H}$).

Note that (2.9) follows when $\mathbf{E}$ is any of the material strains in the class of measures presented by Hill[50]. A brief outline of this and related results is given in Section 3.

Let $\mathscr{L}$ be the instantaneous (Eulerian) elastic modulus tensor, the inverse of $\mathscr{M}$. Equation (2.9) then becomes

$$\overset{\circ}{\boldsymbol{\tau}} = \mathscr{L}:\mathbf{D}^e. \tag{2.12}$$

Both $\mathscr{L}$ and $\mathscr{M}$ change with the choice of stress rate. The difference, $\mathscr{L}-\hat{\mathscr{L}}$, is a linear function of the Cauchy stress, as discussed by Hill[51], where $\hat{\mathscr{L}}$ is the modulus tensor associated with another objective stress rate measured with respect to the current configuration.

In many cases of practical importance, the dependency of, say, the strain $\mathbf{E}$ on the history $\mathbf{H}$ may be such that the difference eqn (2.7) can be represented in terms of partial differentiation, Mandel[39], Rice[52] and Nemat-Nasser[53],

$$d^p\mathbf{E} = \frac{\partial \mathbf{E}}{\partial H_\alpha}\,\mathrm{d}H_\alpha, \tag{2.13}$$

where α is summed over all parameters (internal variables, e.g. the active slip systems) that are responsible for inducing the *plastic* incremental strain. In such situations we write, instead of (2.5),

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}^e + \mathbf{E}^p. \tag{2.14}$$

Furthermore, if the current configuration is used as the reference one, it follows that

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p. \tag{2.15}$$

Single crystals are often described in terms of constitutive relations that fall in the general class outlined above, as discussed in Section 1. Specifically, if there are N active slip planes with orientation $\mathbf{n}^\alpha$ and slip direction $\mathbf{s}^\alpha$, and when the corresponding rates of slip are $\dot{\gamma}^\alpha$, it follows that

$$\mathbf{D}^p + \mathbf{W}^p = \sum_{\alpha=1}^{N} \mathbf{s}^\alpha \otimes \mathbf{n}^\alpha \dot{\gamma}^\alpha, \tag{2.16}$$

where $\mathbf{W}^p$ is the slip-induced spin, and $\otimes$ denotes tensor product; $\mathbf{n}^\alpha$ and $\mathbf{s}^\alpha$ are unit vectors. The total velocity gradient measured with respect to the current configuration then is

$$\begin{aligned} \mathbf{L} &= \mathbf{D} + \mathbf{W} \\ &= \{\mathbf{D}^e + \mathbf{W}^*\} + \{\mathbf{D}^p + \mathbf{W}^p\}, \end{aligned} \tag{2.17}$$

where $\mathbf{W}^*$ is the lattice spin.

In the summary presented above, the history parameter $\mathbf{H}$ may be interpreted in a variety of ways, and therefore the results apply to a large class of materials; the corresponding material description encompasses essentially all existing plasticity theories. The reference configuration and the material strain measure may be selected as desired. Hence, the additive decompositions (2.5), (2.14), (2.15) and (2.17), or their equivalents associated with other strain measures relative to other ground states, hold rigorously with one proviso: the *same* strain measure referred to the same reference state must be used† for the total, the elastic and the plastic strain rates.

†As is shown in Section 4, Lee[33] defines an elastic strain rate, $\mathbf{D}_L^e$, with respect to the current configuration, and a plastic strain rate, $\mathbf{D}_L^p$, with respect to *another* intermediate configuration; naturally, these cannot add to yield $\mathbf{D}$, the Eulerian rate.

2.1 *Invariance of rate of stress work*

The question then arises, how one decides whether given elastic and plastic strain rate measures for the same rate of deformation, *correspond* to each other in the sense of the above proviso. The answer is obtained from the invariance of the rate of stress work.

The quantity $1/\rho_0 \operatorname{tr}(\boldsymbol{\tau}\mathbf{D})$ is the rate of work per unit mass and therefore is invariant under the change of reference state and strain measures where ρ_0 is the reference density. From (2.15), it follows that

$$\frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}) = \frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}^e) + \frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}^p). \tag{2.18}$$

The first term in the r.h.s. is the rate of elastic energy (e.g. due to lattice distortion), and the second term is the rate of plastic dissipation (e.g. due to slip over crystallographic planes), both per unit mass. *Each* of these physical quantities remains invariant under the change of reference state or the strain measure, and the two quantities must always add up to the total rate of stress work per unit mass, $1/\rho_0 \operatorname{tr}(\boldsymbol{\tau}\mathbf{D})$: conservation of energy. For example,

$$\begin{aligned}\frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}) &= \frac{1}{\rho_0}\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}) = \frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}^e) + \frac{1}{\rho_0}\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}^p)\\ &= \frac{1}{\rho_0}\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}^e) + \frac{1}{\rho_0}\operatorname{re}(\boldsymbol{\tau}\mathbf{D}^p)\\ &= \frac{1}{\rho_0}\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}^e) + \frac{1}{\rho_0}\operatorname{tr}(\mathbf{S}\dot{\mathbf{E}}^p).\end{aligned} \tag{2.19}$$

2.2 *Rate constitutive relations*

In the final constitutive relations, it is, for example, the Jaumann rate of, say, Kirchhoff stress, corotational with the *total* material spin, which must be related to the corresponding *total* Eulerian strain rate. To this end, set

$$\overset{\nabla}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{W} \tag{2.20}$$

and from (2.10), (2.12) and (2.17) obtain

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathscr{L}\colon \mathbf{D} - \{\mathscr{L}\colon \mathbf{D}^p + \mathbf{W}^p\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W}^p - \boldsymbol{\tau}\operatorname{tr}\mathbf{D}^p\}. \tag{2.21}$$

For single crystals, for example, $\mathbf{D}^p$ and $\mathbf{W}^p$ are the symmetric and antisymmetric parts of the r.h.s. of (2.16). In a macroscopic approach, on the other hand, $\mathbf{D}^p$ and $\mathbf{W}^p$ are defined phenomenologically, often ignoring the terms associated with $\mathbf{W}^p$ and $\operatorname{tr}\mathbf{D}^p$ in (2.21).

3. CHOICE OF STRAIN MEASURES AND GROUND STATES

Much of the confusion in finite elasto-plasticity seems to stem from the particular choice of strain measures and reference states. While such choices do alter the form of the quantities involved, they do not affect the basic physical ingredients that are governed by certain fundamental invariance principles. Although this and related facts have been extensively developed by Hill, e.g. [49, 50], and recently presented in a unified manner [40], it may be useful to summarize here a few relevant items, in order to further emphasize that the validity of the general results in Section 2 does not depend on the particular choice of strain measure or reference state.

To this end, let $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ with Cartesian components $F_{iA} = \partial x_i/\partial X_A$, $i, A = 1, 2, 3$, be the deformation gradient of the one-to-one mapping, $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, of material elements from the undeformed initial configuration, $\mathscr{C}_0$, to the elasto-plastically deformed current configuration, $\mathscr{C}$. By the polar decomposition,

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{3.1}$$

where $\mathbf{R}$ is the rotation, $\mathbf{R}^{-1} = \mathbf{R}^T$, $\det \mathbf{R} = 1$, and the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$, with principal values $\lambda_{(a)}$ and the respective principal directions $\mathbf{N}_a$ and $\mathbf{n}_a$, $a = 1, 2, 3$, are symmetric and positive-definite, where $\mathbf{n}_a = \mathbf{R}\mathbf{N}_a$. With respect to the principal triads (the Lagrangian, $\mathbf{N}_a$, and the Eulerian, $\mathbf{n}_a$), we record the following well-known representations:

$$\begin{aligned} \mathbf{F} &= \sum_{a=1}^{3} \lambda_{(a)} \mathbf{n}_a \otimes \mathbf{N}_a, \quad \mathbf{R} = \sum_{a=1}^{3} \mathbf{n}_a \otimes \mathbf{N}_a, \\ \mathbf{U} &= \sum_{a=1}^{3} \lambda_{(a)} \mathbf{N}_a \otimes \mathbf{N}_a, \quad \mathbf{V} = \sum_{a=1}^{3} \lambda_{(a)} \mathbf{n}_a \otimes \mathbf{n}_a. \end{aligned} \tag{3.2}$$

From $\mathbf{u}$ and $\mathbf{V}$, other, respectively Lagrangian (material) and Eulerian, deformation measures are formed. In particular, a general class of material strain measures is, Hill [50],

$$\mathbf{E} = \sum_{a=1}^{3} f(\lambda_{(a)}) \mathbf{N}_a \otimes \mathbf{N}_a, \tag{3.3}$$

where $f(1) = 0$, $f'(1) = 1$ and $f'(\lambda) > 0$; a subclass which includes most commonly used measures, corresponds to $f(\lambda) = (\lambda^{2m} - 1)/2m$, see Seth [54] and Hill [50]. Associated with (3.3), we have the Eulerian measure

$$\mathbf{e} = \sum_{a=1}^{3} f(\lambda_{(a)}) \mathbf{n}_a \otimes \mathbf{n}_a = \mathbf{R}\mathbf{E}\mathbf{R}^T. \tag{3.4}$$

Also, from (3.2), the usual right, $\mathbf{C}$, and left, $\mathbf{B}$, Cauchy–Green tensors are

$$\begin{aligned} \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 &= \sum_{a=1}^{3} \lambda_{(a)}^2 \mathbf{N}_a \otimes \mathbf{N}_a, \\ \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 &= \sum_{a=1}^{3} \lambda_{(a)}^2 \mathbf{n}_a \otimes \mathbf{n}_a, \\ \mathbf{C} = \mathbf{R}^T\mathbf{B}\mathbf{R}, \quad \mathbf{B} &= \mathbf{R}\mathbf{C}\mathbf{R}^T. \end{aligned} \tag{3.5}$$

Finally, note that the usual logarithmic strain, $\ln \mathbf{U}$, is obtained from (3.3) by simply setting $f(\lambda) = \ln \lambda$.

3.1 *Deformation rates and their decomposition*

Lagrangian and Eulerian deformation rates are obtained by material-time differentiation of (3.3) and (3.4), respectively,

$$\dot{\mathbf{E}} = \sum_{a,b=1}^{3} \{f'(\lambda_{(a)})\dot{\lambda}_a \delta_{ab} + [f(\lambda_{(b)}) - f(\lambda_{(a)})]\Omega^L_{(ab)}\} \mathbf{N}_a \otimes \mathbf{N}_b \tag{3.6}$$

$$\dot{\mathbf{e}} = \sum_{a,b=1}^{3} \{f'(\lambda_{(a)})\dot{\lambda}_{(a)} \delta_{ab} + [f(\lambda_{(b)}) - f(\lambda_{(a)})]\Omega^E_{(ab)}\} \mathbf{n}_a \otimes \mathbf{n}_b \tag{3.7}$$

where

$$\boldsymbol{\Omega}^L = \sum_{a,b=1}^{3} \Omega^L_{(ab)} \mathbf{N}_a \otimes \mathbf{N}_b \quad \text{and} \quad \boldsymbol{\Omega}^E = \sum_{a,b=1}^{3} \Omega^E_{(ab)} \mathbf{n}_a \otimes \mathbf{n}_b \tag{3.8}$$

are the spin of the Lagrangian and Eulerian triads, respectively; $\dot{\mathbf{N}}_a = \boldsymbol{\Omega}^L \mathbf{N}_a$ and $\dot{\mathbf{n}}_a = \boldsymbol{\Omega}^E \mathbf{n}_a$. Here and in the sequel, components with respect to the principal triads are denoted by indices within parentheses. Let the spin of the $\mathbf{n}_a$-triad relative to the $\mathbf{N}_a$-triad be denoted by $\boldsymbol{\Omega}^R$,

$$\dot{\mathbf{R}} = \boldsymbol{\Omega}^R \mathbf{R}, \quad \boldsymbol{\Omega}^R = \sum_{a,b=1}^{3} \Omega^R_{(ab)} \mathbf{n}_a \otimes \mathbf{n}_b, \tag{3.9}$$

and express the deformation rate tensor **D** and the spin tensor **W** as

$$\mathbf{D} = \sum_{a,b=1}^{3} D_{(ab)}\mathbf{n}_a \otimes \mathbf{n}_b, \quad \mathbf{W} = \sum_{a,b=1}^{3} W_{(ab)}\mathbf{n}_a \otimes \mathbf{n}_b. \tag{3.10}$$

Then, it can be shown that, Hill[40, 55],

$$\begin{aligned}
&\Omega^L_{(ab)} = \Omega^E_{(ab)} - \Omega^R_{(ab)} \\
&\Omega^L_{(ab)} = \frac{2\lambda_{(a)}\lambda_{(b)}}{\lambda^2_{(b)} - \lambda^2_{(a)}} D_{(ab)}, \quad a \neq b, \quad \lambda_{(a)} \neq \lambda_{(b)}, \\
&\Omega^E_{(ab)} = \frac{\lambda^2_{(b)} + \lambda^2_{(a)}}{\lambda^2_{(b)} - \lambda^2_{(a)}} D_{(ab)} + W_{(ab)}, \quad a \neq b, \quad \lambda_{(a)} \neq \lambda_{(b)}.
\end{aligned} \tag{3.11}$$

From (3.11) it follows that the class of material strain rates defined by (3.6) is objective, whereas the Eulerian one defined by (3.7) involves the material spin, **W**. Furthermore, writing

$$\dot{\mathbf{E}} = \sum_{a,b=1}^{3} \dot{E}_{(ab)}\mathbf{N}_a \otimes \mathbf{N}_b \tag{3.12}$$

and since $D_{(aa)} = \dot{\lambda}_{(a)}/\lambda_{(a)}$ (no sum on a), one obtains

$$\begin{aligned}
&\dot{E}_{(aa)} = \lambda_{(a)} f'(\lambda_{(a)}) D_{(aa)} \quad \text{(no sum on } a\text{)}, \\
&\dot{E}_{(ab)} = [f(\lambda_{(b)}) - f(\lambda_{(a)})] \frac{2\lambda_{(a)}\lambda_{(b)}}{\lambda^2_{(b)} - \lambda^2_{(a)}} D_{(ab)}, \quad a \neq b, \quad \lambda_{(a)} \neq \lambda_{(b)},
\end{aligned} \tag{3.13}$$

so that the decomposition

$$D_{(ab)} = D^e_{(ab)} + D^p_{(ab)}$$

leads to the decomposition

$$\dot{E}_{(ab)} = \dot{E}^e_{(ab)} + \dot{E}^p_{(ab)} \tag{3.14}$$

for the general class of Lagrangian strain rates (3.6). Also, since $W_{(ab)} = W^*_{(ab)} + W^p_{(ab)}$, where $W^*_{(ab)}$ is the elastically induced spin, a similar additive decomposition applies to $\dot{\mathbf{e}}$ of eqn (3.7).

In the same manner, direct material-time differentiation of (3.5) yields, in view of (3.11),

$$\dot{\mathbf{C}} = \sum_{a,b=1}^{3} 2\lambda_{(a)}\lambda_{(b)} D_{(ab)}\mathbf{N}_a \otimes \mathbf{N}_b, \tag{3.15}$$

$$\dot{\mathbf{B}} = \sum_{a,b=1}^{3} \{(\lambda^2_{(b)} + \lambda^2_{(a)}) D_{(ab)} + (\lambda^2_{(b)} - \lambda^2_{(a)}) W_{(ab)}\}\mathbf{n}_a \otimes \mathbf{n}_b, \tag{3.16}$$

so that these rates also admit additive decompositions,

$$\dot{\mathbf{C}} = \dot{\mathbf{C}}^e + \dot{\mathbf{C}}^p, \quad \dot{\mathbf{B}} = \dot{\mathbf{B}}^e + \dot{\mathbf{B}}^p, \tag{3.17}$$

where the "elastic" and "plastic" contributions are obtained by substitution of the elastic and plastic parts of **D** and **W** into eqns (3.15) and (3.16). Observe that the rate

$$\dot{\mathbf{B}} - \boldsymbol{\omega} = \sum_{a,b=1}^{3} (\lambda^2_{(a)} + \lambda^2_{(b)}) D_{(ab)}\mathbf{n}_a \otimes \mathbf{n}_b \tag{3.18}$$

is objective in the sense that it does not involve the material spin **W**, where

$$\boldsymbol{\omega} = \sum_{a,b=1}^{3} (\lambda^2_{(b)} - \lambda^2_{(a)}) W_{(ab)}\mathbf{n}_a \otimes \mathbf{n}_b. \tag{3.19}$$

The stress measures conjugate to the general class of strain measures (3.3) are easily obtained from the invariance of the rate of stress work per unit mass, $1/\rho_0 \operatorname{tr}(\boldsymbol{\tau}\mathbf{D}) = 1/\rho_0 \operatorname{tr}(\mathbf{S}\dot{\mathbf{E}})$, where $\mathbf{S}$ stands for the stress which is conjugate to the considered strain $\mathbf{E}$. In the Lagrangian triad,

$$\mathbf{S} = \sum_{a,b=1}^{3} S_{(ab)} \mathbf{N}_a \otimes \mathbf{N}_b, \tag{3.20}$$

and in the Eulerian triad

$$\boldsymbol{\tau} = \sum_{a,b=1}^{3} \tau_{(ab)} \mathbf{n}_a \otimes \mathbf{n}_b, \tag{3.21}$$

so that, in general,

$$\begin{aligned} S_{(aa)} &= \frac{\tau_{(aa)}}{\lambda_{(a)} f'(\lambda_{(a)})} \text{ (no sum on } a), \\ S_{(ab)} &= \frac{\lambda_{(b)}^2 - \lambda_{(a)}^2}{f(\lambda_{(b)}) - f(\lambda_{(a)})} \frac{\tau_{(ab)}}{2\lambda_{(a)}\lambda_{(b)}}, \quad a \neq b, \quad \lambda_{(a)} \neq \lambda_{(b)}. \end{aligned} \tag{3.22}$$

Note that for the logarithmic strain, $\ln \mathbf{U}$, $S^l_{(aa)} = \tau_{(aa)}$ (no sum on a) and $S^l_{(ab)} = \tau_{(ab)}\{(\lambda_{(b)}^2 - \lambda_{(a)}^2\}/\{2\lambda_{(a)}\lambda_{(b)} \ln(\lambda_{(b)}/\lambda_{(a)})\}$.

It has been observed by Hill, e.g. [12, 40], see also Havner[29], that many aspects of polycrystalline solids and composites are effectively described in terms of the rate of change of the deformation gradient, $\dot{\mathbf{F}}$, and its conjugate stress, $\boldsymbol{\delta}$, i.e. the nominal stress. The invariance of the rate of stress work then becomes,

$$\begin{aligned} \frac{1}{\rho_0} \operatorname{tr}(\boldsymbol{\tau}\mathbf{D}) &= \frac{1}{\rho_0} \operatorname{tr}(\boldsymbol{\delta}\dot{\mathbf{F}}) \\ &= \frac{1}{\rho_0} \delta_{Ai} \dot{F}_{iA}, \end{aligned} \tag{3.23}$$

where

$$\boldsymbol{\tau} = \mathbf{F}\boldsymbol{\delta} \quad \text{or} \quad \tau_{ij} = F_{iA}\delta_{Aj} \tag{3.24}$$

defines the nominal stress $\boldsymbol{\delta}$ with components δ_{Ai}, $A, i = 1, 2, 3$. Equations (3.23) and (3.24) will be used in the sequel.

3.2 *Change of ground state*

Suppose a new reference (intermediate) configuration $\mathscr{C}_p$ of mass density $\hat{\rho}$ is used, where the deformation gradient from the initial $\mathscr{C}_0$ to $\mathscr{C}_p$ is $\mathbf{F}^p$ with Cartesian components $F^p_{\alpha A}$, and that from $\mathscr{C}_p$ to the current $\mathscr{C}$ is $\mathbf{F}^e$ with Cartesian components $F^e_{i\alpha}$. (Here, the superposed e and p are *not* interpreted as "elastic" and "plastic" parts, as such special and basically questionable identification has no bearing on the general results that are at focus. Later on the physical implications of possible elastic–plastic decomposition of $\mathbf{F}$ into $\mathbf{F}^e$ and $\mathbf{F}^p$ are discussed; see Section 4.) It now follows that

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \quad \text{or} \quad F_{iA} = F^e_{i\alpha}F^p_{\alpha A}, \tag{3.25}$$

and in view of (3.5) we obtain

$$\mathbf{C} = \mathbf{F}^{pT}\hat{\mathbf{C}}^e\mathbf{F}^p \quad \text{or} \quad C_{AB} = F^p_{\alpha A}F^p_{\beta B}\hat{C}^e_{\alpha\beta}, \tag{3.26}$$

$$\mathbf{B} = \mathbf{F}^e\hat{\mathbf{B}}^p\mathbf{F}^{eT} \quad \text{or} \quad B_{ij} = F^e_{i\alpha}F^e_{j\beta}\hat{B}^p_{\alpha\beta}, \tag{3.27}$$

where

$$\hat{\mathbf{C}}^e = \mathbf{F}^{eT}\mathbf{F}^e \quad \text{or} \quad \hat{C}^e_{\alpha\beta} = F^e_{i\alpha}F^e_{i\beta} \tag{3.28}$$

and

$$\hat{\mathbf{B}}^p = \mathbf{F}^p\mathbf{F}^{pT} \quad \text{or} \quad \hat{B}^p_{\alpha\beta} = F^p_{\alpha A}F^p_{\beta A}. \tag{3.29}$$

In these equations, $\hat{\mathbf{C}}^e$ is the *right* Cauchy–Green tensor for mapping from the *intermediate* configuration $\mathscr{C}_p$ to the *current* one $\mathscr{C}$, whereas $\hat{\mathbf{B}}^p$ is the *left* Cauchy–Green tensor for mapping from the *initial* $\mathscr{C}_0$ to the *intermediate* $\mathscr{C}_p$ configuration. Both $\hat{\mathbf{C}}^e$ and $\hat{\mathbf{B}}^p$ *are referred to the intermediate configuration.* Equations (3.26) and (3.27), therefore, are the standard results for transformation of second order tensors.

From (3.25), we have

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e\mathbf{F}^p + \mathbf{F}^e\dot{\mathbf{F}}^p \tag{3.30}$$

and therefore, the rate of stress work becomes

$$\begin{aligned}\frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\tau}\mathbf{D}) &= \frac{1}{\rho_0}\operatorname{tr}(\boldsymbol{\delta}\dot{\mathbf{F}}) \\ &= \frac{1}{\hat{\rho}}\operatorname{tr}(\hat{\boldsymbol{\delta}}\dot{\mathbf{F}}^e) + \frac{1}{\rho_0}\operatorname{tr}(\hat{\boldsymbol{\delta}}\dot{\mathbf{F}}^p),\end{aligned} \tag{3.31}$$

where

$$\hat{\boldsymbol{\delta}} = \frac{\hat{\rho}}{\rho_0}\mathbf{F}^p\boldsymbol{\delta} \quad \text{or} \quad \hat{\delta}_{\alpha i} = \frac{\hat{\rho}}{\rho_0}F^p_{\alpha A}\delta_{Ai} \tag{3.32}$$

is the nominal stress transformed to the intermediate state, and

$$\hat{\boldsymbol{\delta}} = \boldsymbol{\delta}\mathbf{F}^e \quad \text{or} \quad \hat{\delta}_{A\alpha} = \delta_{Ai}F^e_{i\alpha} \tag{3.33}$$

is the stress conjugate to $\dot{\mathbf{F}}^p$. Note that $\hat{\delta}_{\alpha i}$ is the ith component of the traction vector transmitted (and measured per unit area in the intermediate state) across an element which in the intermediate state is normal to the αth direction; no such simple interpretation can be given to $\hat{\delta}_{A\alpha}$.

Consider now the rate of change of $\mathbf{C}$ and $\mathbf{B}$, and obtain

$$\dot{\mathbf{C}} = \mathbf{F}^{pT}\dot{\hat{\mathbf{C}}}^e\mathbf{F}^p + \dot{\mathbf{F}}^{pT}\hat{\mathbf{C}}^e\mathbf{F}^p + \mathbf{F}^{pT}\hat{\mathbf{C}}^e\dot{\mathbf{F}}^p, \tag{3.34}$$

$$\dot{\mathbf{B}} = \mathbf{F}^e\dot{\hat{\mathbf{B}}}^p\mathbf{F}^{eT} + \dot{\mathbf{F}}^e\hat{\mathbf{B}}^p\mathbf{F}^{eT} + \mathbf{F}^e\hat{\mathbf{B}}^p\dot{\mathbf{F}}^{eT}. \tag{3.35}$$

In (3.34), the first term in the r.h.s. is $\dot{\hat{\mathbf{C}}}^e$ transformed to the *initial* state, whereas in the second two terms $\hat{\mathbf{C}}^e$ acts as the *metric* tensor. In a similar manner, in (3.35), the first term in the r.h.s. is $\dot{\hat{\mathbf{B}}}^p$ transformed to the *current* state, while in the second two terms $\hat{\mathbf{B}}^p$ acts as the metric tensor. Since $\hat{\mathbf{C}}^e$ and $\hat{\mathbf{B}}^p$ *are independent of rates*, one writes

$$\dot{\mathbf{C}} = \dot{\mathbf{C}}^e + \dot{\mathbf{C}}^p \quad \text{and} \quad \dot{\mathbf{B}} = \dot{\mathbf{B}}^e + \dot{\mathbf{B}}^p, \tag{3.36}$$

where

$$\dot{\mathbf{C}}^e = \mathbf{F}^{pT}\dot{\hat{\mathbf{C}}}^e\mathbf{F}^p, \quad \dot{\mathbf{C}}^p = \dot{\mathbf{F}}^{pT}\hat{\mathbf{C}}^e\mathbf{F}^p + \mathbf{F}^{pT}\hat{\mathbf{C}}^e\dot{\mathbf{F}}^p, \tag{3.37}$$

$$\dot{\mathbf{B}}^p = \mathbf{F}^e\dot{\hat{\mathbf{B}}}^p\mathbf{F}^{eT}, \quad \dot{\mathbf{B}}^e = \dot{\mathbf{F}}^e\hat{\mathbf{B}}^p\mathbf{F}^{eT} + \mathbf{F}^e\hat{\mathbf{B}}^p\dot{\mathbf{F}}^{eT}. \tag{3.38}$$

These interpretations are valid independently of the particular physical nature of the decomposition (3.25); that is, *in general, the rates decompose additively for a sequence of finite deformations which lead from an initial configuration to a current one.* The rate-independent (matrices) factors in (3.37) and (3.38) are simply the *normalizing* parameters which are required in order to refer the involved physical quantities to a *common* reference state so that they become *comparable* quantities.

4. A CONFLICTING THEORY BY E. H. LEE[33, 36, 37]

Lee and his associate[33, 36, 37] have proposed an elasto-plasticity theory for finite deformations, which, at least according to their interpretation, conflicts with the general results presented in this paper. It is therefore necessary to examine various details of their approach, in order to bring into focus the major points responsible for the apparent conflict.

Lee's approach[33] is based on the *total purely elastic* unloading from the current state $\mathscr{C}$, to an intermediate *unstressed* plastically deformed configuration $\mathscr{C}_p$, without *any* reverse or other kind of plastic flow.† The major point in the theory is to decouple the total elastically induced distortion and measure it from a relaxed unstressed state which is only plastically deformed from the initial $\mathscr{C}_0$ to the intermediate $\mathscr{C}_p$ configuration.

As was pointed out, plastic flow occurs by slip over crystallographic planes on active slip systems at microlevel, and microscopically a crystalline solid consists of randomly distributed grains that contain precipitates and second-phase inclusions, and, therefore, is strongly heterogeneous; it is the random distribution of micro-heterogeneities that results in a macroscopically homogeneous response. Therefore, after a fair amount of plastic flow has taken place, reverse plastic deformation will result soon upon unloading,‡ and, hence a total elastic unloading cannot have any physical significance. It is purely conceptual, and should be viewed as such. In [33] Lee presents an argument in support of such total elastic unloading, suggesting that cuts must be made, in order to release the elastic distortion without additional plastic flow. Since plastic flow is at crystal levels and involves the motion of dislocations, such cuts are required at the same scale which no longer falls in the continuum realm.

Notwithstanding these rather serious shortcomings of the basic hypothesis in Lee's theory, one may seek to find out if it still can lead to useful theoretical developments. With this in mind, therefore, let us cast this theory in the general framework presented in Section 2, and then examine its various consequences.

To this end, assume that it is possible to implement (at least conceptually) a total elastic unloading associated with deformation gradient $\mathbf{F}^e$, without additional plastic flows. In the terminology of Hill and Rice[38] and Mandel[39] this means that the stress is reduced to zero, while the history $\mathbf{H}$ is kept fixed (e.g. all mobile or potentially mobile dislocations, and all active or potentially active slip systems are "locked"). The intermediate configuration $\mathscr{C}_p$ then depends only on the history $\mathbf{H}$ and not on the elastic distortion which produces the stress, say, $\mathbf{S}$. Thus, considering a specific material element with deformation gradient $\mathbf{F} = \mathbf{F}(\mathbf{S}, \mathbf{H})$, we have the decomposition (3.25), i.e. $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, in which

$$\mathbf{F}^p = \mathbf{F}^p(\mathbf{H}), \quad \mathbf{F}^e = \mathbf{F}^e(\mathbf{S}, \mathbf{H}), \tag{4.1}$$

where the dependence on the particular material element is suppressed.

Lee argues that $\mathbf{F}^e$ does not depend on the history. His argument is based on some experimental observations that metal elasticity is not changed much because of prior plastic flow. While this assumption has a limited applicability, there is extensive experimental evidence which shows that strong anisotropy, for example, develops upon large plastic deformations. This is true in the case of polycrystalline solids, where textures can form by large rotations and extensive plastic flow of grains which can actually attain different geometric shapes during plastic flows. This is a common occurrence in wire-drawing and sheet-metal forming. The grains become elongated in the direction of flow, and finite total lattice rotations biased in specific

†In a more recent article[36] Lee seems to have changed his view on this point, as he now apparently admits reverse plastic flow, but seems to suggest that this does not have importance in his theory. The arguments are rather confused, and seem to be contradictory to his other earlier works[33].

‡Indeed Hutchinson[22] shows for polycrystalline solids that even for small strains reverse plastic flow may start soon after unloading from an advanced stage of elasto-plastic deformation.

directions result in strong elastic anisotropy, as well as in a plastic one. Furthermore, voids initiate at second-phase particles and grow during the course of plastic deformation, and depending on the shape and distribution of these voids, the overall elasticity of the solid is changed considerably. Therefore, it does not seem reasonable to assume that the elastic response, even if it could be decoupled from the plastic one, in the manner shown by eqn (3.25), remains history-independent.

Lee's entire argument, on the other hand, is based on the assumption that prior plastic flow leaves the elastic response essentially unchanged. Based on this assumption, he then introduces, from (3.30),

$$\begin{aligned}\mathbf{L} \equiv \dot{\mathbf{F}}\mathbf{F}^{-1} &= \dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}} + \mathbf{F}^e\{\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}\}\mathbf{F}^{e^{-1}} \\ &= \mathbf{L}_L{}^e + \mathbf{F}^e\mathbf{L}_L{}^p\mathbf{F}^{e^{-1}}\end{aligned} \tag{4.2}$$

where

$$\mathbf{L}_L{}^e = \dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}} \tag{4.3}$$

according to Lee, "corresponds to the velocity gradient of the purely elastic deformation", and

$$\mathbf{L}_L{}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}} \tag{4.4}$$

"corresponds to the velocity gradient of the purely plastic deformation"; in eqns (4.2)–(4.4), the subscript L is added, in order to identify the definition with Lee.

In view of the definitions (4.3) and (4.4), Lee then concludes that the sum of (4.3) and (4.4) does not add to give $\mathbf{L}$, and then he claims that this shows that "the total strain rate is not equal to the sum of elastic and plastic rates, as is universally assumed".

Naturally, if the plastic contribution to the total velocity gradient is *defined* by (4.4), the additive decomposition of strain rates applies only if $\mathbf{F}^e \approx \mathbf{I}$, the identity tensor. However, it is not clear why one should insist on definition (4.4), even *if* the total decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ does actually decouple elasticity from plasticity. Since $\mathbf{F}^e$ is *rate-independent*, it serves as a *normalizing* factor in the expression

$$\mathbf{L}^p \equiv \mathbf{F}^e\mathbf{L}_L{}^p\mathbf{F}^{e^{-1}}, \tag{4.5}$$

and therefore, this expression is an equally (but not more) acceptable definition for the plastic contribution to the velocity gradient. In fact, as discussed in the preceding section, the particular choice of the strain measures does not affect the physical content, and should not have any bearing on the final conclusions.

From (4.2) Lee concludes that additivity applies when the *elastic* deformation gradient is almost identity.

By considering the left Cauchy–Green tensor, $\mathbf{B}$, one can use Lee's argument and equally conclude that additivity applies when the *plastic* part of the deformation gradient, $\mathbf{F}^p$, is almost identity.

Naturally, neither argument applies. Nevertheless, it is constructive to produce two parallel formulations based on the right and left Cauchy–Green tensors, as in Section 3, and examine the consequences.

These are given in eqns (3.34) and (3.35). In eqn (3.34), for example, $\mathring{\mathbf{C}}^e$ corresponds precisely to the symmetric part of $\mathbf{L}_L{}^e$. Indeed, if we set

$$\mathbf{D}_L{}^e = \frac{1}{2}(\mathbf{L}_L{}^e + \mathbf{L}_L{}^{eT}), \tag{4.6}$$

and then transform to the *initial* configuration, we obtain

$$\begin{aligned}2\mathbf{F}^T\mathbf{D}_L{}^e\mathbf{F} &= \mathbf{F}^{pT}\{\dot{\mathbf{F}}^{eT}\mathbf{F}^e + \mathbf{F}^{eT}\dot{\mathbf{F}}^e\}\mathbf{F}^p \\ &= \mathbf{F}^{pT}\mathring{\mathbf{C}}^e\mathbf{F}^p \equiv \dot{\mathbf{C}}^e\end{aligned} \tag{4.7}$$

which is precisely the first term in the r.h.s. of (3.34). A similar interpretation applies to the first term in the r.h.s. of (3.35), but this time it is the plastic (according to Lee's definition) part of the left Cauchy–Green tensor measured with respect to the intermediate configuration that is transformed to the *current* configuration, i.e.

$$\dot{\mathbf{B}}^p = \mathbf{F}^e \dot{\hat{\mathbf{B}}}^p \mathbf{F}^{e^T}, \quad \dot{\hat{\mathbf{B}}}^p = \{\mathbf{F}^p \mathbf{F}^{p^T}\}^{\cdot}$$

Furthermore, the second two terms in the r.h.s. of (3.34) reduce to $\{\mathbf{F}^{p^T}\mathbf{F}^p\}^{\cdot}$, if $\mathbf{F}^e \approx \mathbf{I}$, in which case $\hat{\mathbf{C}}^e \approx \mathbf{I}$. In a similar manner, the second two terms in the r.h.s. of (3.35) reduce to $\{\mathbf{F}^e\mathbf{F}^{e^T}\}^{\cdot}$, if $\mathbf{F}^p \approx \mathbf{I}$, in which case $\hat{\mathbf{B}}^p \approx \mathbf{I}$. While these mathematical deductions always hold, they have no bearing on the additive decompositions given by eqns (3.36) which apply to *any* sequence of two mappings that take the body from the initial to the current configuration through an intermediate state. The factors $\hat{\mathbf{C}}^e$ in the last two terms in the r.h.s. of (3.34), and $\hat{\mathbf{B}}^p$ in the last two terms in the r.h.s. of (3.35) are the corresponding metric tensors which reduce to the identity tensor if the corresponding *total* deformations are suitably small. Surely these rate independent normalizing factors do not alter the physical nature of the strain rates.

4.1 *On the meaning of elastic strain rate*

As pointed out above, even if the decomposition (3.25) is accepted on the conceptual basis, still $\mathbf{F}^e$ cannot be regarded to be independent of the history of the plastic flow. In a recent work[34] I attempted to explain this fact from a purely phenomenological approach, where I showed that the elastic strain rate defined by (4.6) cannot be independent of the plastic change which is portrayed by the history parameter $\mathbf{H}$ in the present work. Indeed, from $(4.1)_2$ it follows that

$$\dot{\mathbf{F}}^e = \frac{\partial \mathbf{F}^e}{\partial S_{AB}} \dot{S}_{AB} + \frac{\partial \mathbf{F}^e}{\partial H_\alpha} \dot{\mathbf{H}}_\alpha, \tag{4.8}$$

where α is summed over all active sources that contribute to inelasticity, and it is assumed that the operation of partial differentiation can be applied.† For a single crystal, for example, $\dot{\mathbf{H}}_\alpha$ represents the rate of slip associated with the active α-system. If such slip does not alter the lattice structure, which is the case in the ideal crystal, then the elastic and plastic strain rates decouple additively, in the manner discussed in Section 2, however, not if Lee's definitions are used. It is clear that, even for the single crystal, the accumulated total slip over all slip planes that have been active during the entire deformation course, may alter $\mathbf{F}^e$, as shown by eqn (4.8).

The only way that the elastic strain rate defined by eqn (4.6) can remain unaffected by any additional change in history-dependency, i.e. in $\mathbf{H}$, is when, instead of (4.1), one *assumes*

$$\mathbf{F}^p = \mathbf{F}^p(\mathbf{H}) \quad \text{and} \quad \mathbf{F}^e = \mathbf{F}^e(\mathbf{S}). \tag{4.9}$$

In this case, *the prior plastic deformation of the solid is regarded to have no effect on its subsequent elastic response. Then, the same elastic strain applied to any unstressed state results in the same elastic stress.* This is a rather simplifying assumption that underlies the entire kinetic development in Lee's theory. In fact, in a more recent paper[36] Lee suggests that even reverse plastic flows in unloading do not affect the free energy and, therefore, the elastic response.

If one accepts eqn $(4.9)_2$ as *the* working hypothesis, then lattice distortion applied to the *initial* undeformed state (involving no plastic distortion; a thought experiment), should produce the same elastic response, but referred to the initial state. Based on this observation, I pointed out in [34] that then one can introduce an equivalent decomposition

$$\mathbf{F} = \bar{\mathbf{F}}^p \bar{\mathbf{F}}^e, \tag{4.10}$$

where, in accordance with assumption (4.9), we observe here that

$$\bar{\mathbf{F}}^p = \bar{\mathbf{F}}^p(\mathbf{H}) \quad \text{and} \quad \bar{\mathbf{F}}^e = \bar{\mathbf{F}}^e(\mathbf{S}). \tag{4.11}$$

†A similar observation has been made by Mandel[35] who also points out some of the limitations in Lee's theory.

I then showed that

$$\bar{\mathbf{F}}^e = \mathbf{F} - \mathbf{F}^p + \mathbf{I}, \tag{4.12}$$

so that the corresponding velocity gradients decompose additively,

$$\dot{\mathbf{F}} = \dot{\bar{\mathbf{F}}}^e + \dot{\mathbf{F}}^p,$$

where all quantities are now referred to the initial configuration.

In [34] I stressed that decomposition (3.25) due to Lee and with his interpretation, as well as my decomposition (4.10) with a similar interpretation, is "formal, and their practical usefulness remains to be established". From the discussion presented here, it should be quite clear that the assumption of the history-independence of $\mathbf{F}^e$ or, equivalently, $\bar{\mathbf{F}}^e$, is so restrictive that it renders both decompositions essentially useless. Furthermore, even within the limited scope, where elastic response may be assumed as almost history-independent, the total decomposition (3.25) [or (4.10)] produces tremendous mathematical complications in the basic equations, with no apparent advantage. Under the force of these compelling facts, one has no choice other than to yield to the usual incremental elasto-plastic formulation which seems to present considerable advantage, as it can account for induced anisotropy, as well as changes in elasticity due to plastically-induced textures, changes in the grain geometries, accumulated finite lattice rotations, void initiation and growth, and other commonly observed microstructural changes. Furthermore, the incremental formulation presented in Section 2 also applies to the plastic flow of geological and geotechnical materials and granular materials which may even consist of collections of rigid granules, in which the change in fabric is responsible for the stress change (no elasticity); see, for example, Christoffersen *et al.*[48], Oda *et al.*[56], Mehrabadi *et al.*[57], Nemat-Nasser[58], and Nemat-Nasser and Tobita[59].

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